4 Matrix Algebra and Applications

4.2 Matrix Multiplication

Matrix Multiplication

Suppose we buy two CDs at \$3 each and four Zip disks at \$5 each. We calculate our total cost by computing the products' price × quantity and adding:

Cost = $3 \times 2 + 5 \times 4$ = \$26.

Let us instead put the prices in a row vector

$P = [3 \quad 5]$ The price matrix

and the quantities purchased in a column vector,

$Q = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$ The quantity matrix

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Matrix Multiplication

Because *P* represents the prices of the items we are purchasing and *Q* represents the quantities, it would be useful if the product *PQ* represented the total cost, a *single number* (which we can think of as a 1 × 1 matrix).

For this to work, *PQ* should be calculated the same way we calculated the total cost:

$$PQ = [3 \quad 5]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = [3 \times 2 + 5 \times 4] = [26].$$

Notice that we obtain the answer by multiplying each entry in *P* (going from left to right) by the corresponding entry in *Q* (going from top to bottom) and then adding the results.

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The Product *Row* × *Column*

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The Product *Row* × *Column*

The **product** AB of a row matrix A and a column matrix B is a 1×1 matrix.

The length of the row in A must match the length of the column in B for the product to be defined.

To find the product, multiply each entry in A (going from left to right) by the corresponding entry in B (going from top to bottom) and then add the results.

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The Product *Row* × *Column*

Visualizing

$$[2 \quad 4 \quad 1]\begin{bmatrix} 2 \\ 10 \\ -1 \end{bmatrix}$$

2×2	$= 4$	Product of first entries $= 4$
4×10	$= 40$	Product of second entries $= 40$
$1 \times (-1)$	$= -1$	Product of third entries $= -1$
	43	Sum of products $= 43$

Quick Example

$$[2 \quad 1]\begin{bmatrix} -3 \\ 1 \end{bmatrix} = [2 \times (-3) + 1 \times 1]$$
$$= [-6 + 1]$$
$$= [-5]$$

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The Product *Row* × *Column*

Note

The row size has to match the column size. This means that, if we have a 1 × 3 row on the left, then the column on the right must be 3 × 1 in order for the product to make sense. For example, the product

$$[a \quad b \quad c]\begin{bmatrix} x \\ y \end{bmatrix}$$

is not defined.

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The Product *Row* × *Column*

Now to the general case of matrix multiplication:

The Product of Two Matrices: General Case

In general for matrices A and B, we can take the product AB only if the number of columns of A equals the number of rows of B (so that we can multiply the rows of A by the columns of B as above).

The product AB is then obtained by taking its $i\,j$th entry to be:

$i\,j$th entry of AB = Row i of A × Column j of B. As defined above

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The Product *Row* × *Column*

Quick Examples

(R stands for row; C stands for column.)

1. $$R_1 \to [2 \quad 0 \quad -1 \quad 3] \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 1 & 1 & -8 \\ 1 & -6 & 0 \\ 0 & 5 & 2 \\ -3 & 8 & 1 \end{bmatrix} \end{array} = [R_1 \times C_1 \quad R_1 \times C_2 \quad R_1 \times C_3]$$

$$= [-7 \quad 21 \quad -15]$$

2. $$\begin{array}{c} R_1 \to \\ R_2 \to \end{array} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \\ \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix} \end{array} = \begin{bmatrix} R_1 \times C_1 & R_1 \times C_2 \\ R_2 \times C_1 & R_2 \times C_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 10 & -2 \end{bmatrix}$$

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The Product *Row* × *Column*

In matrix multiplication we always take

Rows on the left × Columns on the right.

Look at the dimensions in the two Quick Examples.

$$\overset{\text{Match}}{(1 \times \underset{\downarrow}{4})(\underset{\downarrow}{4} \times 3)} \rightarrow 1 \times 3 \qquad \overset{\text{Match}}{(2 \times \underset{\downarrow}{2})(\underset{\downarrow}{2} \times 2)} \rightarrow 2 \times 2$$

The fact that the number of columns in the left-hand matrix equals the number of rows in the right-hand matrix amounts to saying that the middle two numbers must match as above.

The Product *Row* × *Column*

If we “cancel” the middle matching numbers, we are left with the dimensions of the product.

Before continuing with examples, we state the rule for matrix multiplication formally.

The Product *Row* × *Column*

Multiplication of Matrices: Formal Definition

If A is an $m \times n$ matrix and B is an $n \times k$ matrix, then the product AB is the $m \times k$ matrix whose $i\,j$th entry is the product

$$(AB)_{ij} = \underset{\text{Row } i \text{ of } A}{\underset{\downarrow}{}}[a_{i1} \quad a_{i2} \quad a_{i3} \ldots a_{in}] \underset{\text{Column } j \text{ of } B}{} \begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}.$$

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Example 3 – *Matrix Product*

Calculate:

a. $\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -8 \\ 1 & 0 & 0 \\ 0 & 5 & 2 \\ -2 & 8 & -1 \end{bmatrix}$ **b.** $\begin{bmatrix} -3 \\ 1 \end{bmatrix} [2 \quad 1]$

Solution:

a. Before we start the calculation, we check that the dimensions of the matrices match up.

Match

$$\begin{array}{cc} 2 \times 4 & 4 \times 3 \\ \begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & -1 & 2 & -2 \end{bmatrix} & \begin{bmatrix} 1 & 1 & -8 \\ 1 & 0 & 0 \\ 0 & 5 & 2 \\ -2 & 8 & -1 \end{bmatrix} \end{array}$$

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Example 3 – *Solution*

cont'd

The product of the two matrices is defined, and the product will be a 2 × 3 matrix (we remove the matching 4s: (2 × 4)(4 × 3) → 2 × 3).

To calculate the product, we follow the previous prescription:

$$\begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \end{array} \begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & -1 & 2 & -2 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 1 & 1 & -8 \\ 1 & 0 & 0 \\ 0 & 5 & 2 \\ -2 & 8 & -1 \end{bmatrix} \end{array} = \begin{bmatrix} R_1 \times C_1 & R_1 \times C_2 & R_1 \times C_3 \\ R_2 \times C_1 & R_2 \times C_2 & R_2 \times C_3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 21 & -21 \\ 4 & -5 & -2 \end{bmatrix}.$$

Example 3 – *Solution*

cont'd

b. The dimensions of the two matrices given are 2 × 1 and 1 × 2. Because the 1s match, the product is defined, and the result will be a 2 × 2 matrix.

$$\begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \end{array} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \\ \downarrow & \downarrow \end{array} \\ [\,2 \quad 1\,] \end{array} = \begin{bmatrix} R_1 \times C_1 & R_1 \times C_2 \\ R_2 \times C_1 & R_2 \times C_2 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -3 \\ 2 & 1 \end{bmatrix}$$

The Product *Row* × *Column*

Note

In part (a) we *cannot* multiply the matrices in the opposite order—the dimensions do not match.

We say simply that the product in the opposite order is **not defined**. In part (b) we *can* multiply the matrices in the opposite order, but we would get a 1×1 matrix if we did so.

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The Product *Row* × *Column*

Thus, order is important when multiplying matrices. In general, if *AB* is defined, then *BA* need not even be defined.

If *BA* is also defined, it may not have the same dimensions as *AB*. And even if *AB* and *BA* have the same dimensions, they may have different entries.

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The Product *Row* × *Column*

There are very special square matrices of every size: 1 × 1, 2 × 2, 3 × 3, and so on, called the **identity** matrices.

Identity Matrix

The $n \times n$ identity matrix I is the matrix with 1s down the **main diagonal** (the diagonal starting at the top left) and 0s everywhere else.

In symbols,

$I_{ii} = 1$, and

$I_{ij} = 0$ if $i \neq j$.

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The Product *Row* × *Column*

Quick Examples

1. 1 × 1 identity matrix $I = [1]$

2. 2 × 2 identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note

Identity matrices are always square matrices, meaning that they have the same number of rows as columns. There is no such thing, for example, as the "2 × 4 identity matrix."

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Example 6 – *Identity Matrix*

Evaluate the products AI and IA, where $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and I is the 3×3 identity matrix.

Solution:

First notice that A is arbitrary; it could be any 3×3 matrix.

$$AI = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Example 6 – *Solution* cont'd

In both cases, the answer is the matrix A we started with. In symbols,

$AI = A$

and

$IA = A$

no matter which 3×3 matrix A you start with.

Example 6 – *Solution*

cont'd

Now this should remind you of a familiar fact from arithmetic:

$$a \cdot 1 = a$$

and

$$1 \cdot a = a.$$

That is why we call the matrix I the 3×3 *identity* matrix, because it appears to play the same role for 3×3 matrices that the identity 1 does for numbers.

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The Product *Row* × *Column*

We can now add to the list of properties we gave for matrix arithmetic by writing down properties of matrix multiplication.

In stating these properties, we shall assume that all matrix products we write are defined—that is, that the matrices have correctly matching dimensions.

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The Product *Row* × *Column*

Properties of Matrix Addition and Multiplication

If A, B and C are matrices, if O is a zero matrix, and if I is an identity matrix, then the following hold:

$A + (B + C) = (A + B) + C$	*Additive associative law*
$A + B = B + A$	*Additive commutative law*
$A + O = O + A = A$	*Additive identity law*
$A + (-A) = O = (-A) + A$	*Additive inverse law*
$c(A + B) = cA + cB$	*Distributive law*
$(c + d)A = cA + dA$	*Distributive law*

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The Product *Row* × *Column*

$1A = A$	*Scalar unit*
$0A = O$	*Scalar zero*
$A(BC) = (AB)C$	*Multiplicative associative law*
$c(AB) = (cA)B$	*Multiplicative associative law*
$c(dA) = (cd)A$	*Multiplicative associative law*
$AI = IA = A$	*Multiplicative identity law*
$A(B + C) = AB + AC$	*Distributive law*
$(A + B)C = AC + BC$	*Distributive law*
$OA = AO = O$	*Multiplication by zero matrix*

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The Product *Row* × *Column*

Note that we have not included a multiplicative commutative law for matrices, because the equation $AB = BA$ does not hold in general.

In other words, matrix multiplication is *not* exactly like multiplication of numbers.

We should also say a bit more about transposition. Transposition and multiplication have an interesting relationship.

What is the transposition of a matrix?

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The Product *Row* × *Column*

The Transposition of a Matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

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The Product *Row* × *Column*

Properties of Transposition

$$(A + B)^T = A^T + B^T$$

$$(cA)^T = c(A^T)$$

$$(AB)^T = B^T A^T$$

Notice the change in order in the last one. The order is crucial.

The Product *Row* × *Column*

Quick Examples

1. $$\left(\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}\right)^T = \begin{bmatrix} -2 & 1 \\ 10 & -2 \end{bmatrix}^T \qquad (AB)^T$$

$$= \begin{bmatrix} -2 & 10 \\ 1 & -2 \end{bmatrix}$$

2. $$\begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \qquad B^T A^T$$

$$= \begin{bmatrix} -2 & 10 \\ 1 & -2 \end{bmatrix}$$

The Product *Row* × *Column*

These properties give you a glimpse of the field of mathematics known as **abstract algebra**.

Algebraists study operations like these that resemble the operations on numbers but differ in some way, such as the lack of commutativity for multiplication.